Argyres–Seiberg Duality and New SCFT's

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Outline

I Argyres-Seiberg Duality

Philip Argyres & Nathan Seiberg 0711.0054

$$\mathfrak{g}[\{d_i\}] \ \text{w/}\ \mathbf{r} \ \simeq \ \tilde{\mathfrak{g}}[\{\tilde{d}_i\}] \ \text{w/}\ (\tilde{\mathbf{r}} \oplus \text{SCFT}[d:\mathfrak{h}])$$

- LHS: There is a gauge group $\mathfrak{g}$ with matter charged in representations $\mathbf{r}$.

- RHS: There is a rank 1 SCFT with mass dimension of the Coulomb branch moduli d and flavor symmetry $\mathfrak{h}$. Then $\tilde{\mathfrak{g}} \subset \mathfrak{h}$ is gauged with matter charged in representations $\tilde{\mathbf{r}}$.

II Criteria for Duality

Philip Argyres & JRW 0712.2028

- The spectrum of dimensions of Coulomb branch vevs:
  $\{d_i\} \;=\; \{\widetilde{d}_i\} \;\cup\; \{d\}.$

- The flavor symmetry algebras:
  $f \;=\; \widetilde{f} \;\oplus\; \mathsf{H}.$

- The beta function from weakly gauging the flavor symmetry:
  $\mathsf{T}(\mathbf{r}) \;=\; \mathsf{T}(\widetilde{\mathbf{r}}) \;+\; \mathbf{k}_{\mathfrak{h}} \cdot \mathrm{I}_{\mathsf{H}\hookrightarrow\mathfrak{h}}.$

- The number of marginal couplings:
  $2 \cdot \mathsf{T}(\widetilde{\mathfrak{g}}) \;=\; \mathsf{T}(\widetilde{\mathbf{r}}) \;+\; \mathbf{k}_{\mathfrak{h}} \cdot \mathrm{I}_{\widetilde{\mathfrak{g}}\hookrightarrow\mathfrak{h}}.$

Criteria for Duality(cont'd)

- The contribution to the u(1)$_R$ central charge (for the SCFT):
  $3/2 \cdot k_R \;=\; 24 \cdot c \;=\; 4 \cdot (|\mathfrak{g}| \;-\; |\tilde{\mathfrak{g}}|) \;+\; (|\mathbf{r}| \;-\; |\tilde{\mathbf{r}}|).$

- The contribution to the a conformal anomaly (for the SCFT):
  $48 \cdot a \;=\; 10 \cdot (|\mathfrak{g}| \;-\; |\tilde{\mathfrak{g}}|) \;+\; (|\mathbf{r}| \;-\; |\tilde{\mathbf{r}}|).$

- The existence of a global $\mathbb{Z}_2$-obstruction to gauging the flavor symmetry.

Criteria for Duality (a and c anomalies)

- In Lagrangian theories, the a and c anomalies can be computed by t' Hooft anomaly matching:
  $4 \cdot (2 \cdot a \ - \ c) \ = \ |\mathfrak{g}| \ = \ \sum_i (2 \cdot d_i \ - \ 1).$

- If we look back at the criteria the SCFT satisfies a similar relation:
  $4 \cdot (2 \cdot a \ - \ c) \ = \ (2 \cdot d \ - \ 1).$

- Shapere and Tachikawa have given a proof that this formula holds for a large class of $N = 2$ SCFT's in 0804.1957.

Criteria for Duality ($\mathbb{Z}_2$-obstruction)

E. Witten, An su(2) Anomaly, Phys.Lett.B117:324-3278,1982

G_2 w/ $8 \cdot \mathbf{7} \simeq$ su(2) w/ ($\mathbf{2} \oplus$ SCFT[6 : sp(5)])

- LHS: The $\mathbf{7}$ of G_2 is real $\Rightarrow$ the flavor symmetry is sp(4) when the sp(4) is gauged there is a $\mathbb{Z}_2$-obstruction because the $\mathbf{8}$ is pseudoreal.

- RHS:The su(2) has an anomaly related to the single $\mathbf{2} \Rightarrow$ sp(5) must posses a $\mathbb{Z}_2$-obstruction to gauging in order to cancel this since the LHS is anomaly free $\Rightarrow$ this gives sp(4) a $\mathbb{Z}_2$-obstruction since $\text{I}_{f \hookrightarrow \text{sp}(5)} = 1$ for f either su(2) or sp(4).

III Examples of Duality and Results (1 Marginal Operator)

	$\mathfrak{g}$	w/ $\mathbf{r}$	$=$	$\widetilde{\mathfrak{g}}$	w/ $\widetilde{\mathbf{r}}$	$\oplus$ SCFT $[d:\mathfrak{h}]$
1.	sp(3)	$\mathbf{14} \oplus 11 \cdot \mathbf{6}$	$=$	sp(2)		$[6 : E_8]$
2.	su(6)	$\mathbf{20} \oplus \mathbf{15} \oplus \overline{\mathbf{15}} \oplus 5 \cdot \mathbf{6} \oplus 5 \cdot \overline{\mathbf{6}}$	$=$	su(5)	$\mathbf{5} \oplus \overline{\mathbf{5}} \oplus \mathbf{10} \oplus \overline{\mathbf{10}}$	$[6 : E_8]$
3.	so(12)	$3 \cdot \mathbf{32} \oplus \mathbf{32}' \oplus 4 \cdot \mathbf{12}$	$=$	so(11)	$3 \cdot \mathbf{32}$	$[6 : E_8]$
4.	G_2	$8 \cdot \mathbf{7}$	$=$	su(2)	$\mathbf{2}$	$[6 : \mathrm{sp}(5)]$
5.	so(7)	$4 \cdot \mathbf{8} \oplus 6 \cdot \mathbf{7}$	$=$	sp(2)	$5 \cdot \mathbf{4}$	$[6 : \mathrm{sp}(5)]$
6.	su(6)	$\mathbf{21} \oplus \overline{\mathbf{21}} \oplus \mathbf{20} \oplus \mathbf{6} \oplus \overline{\mathbf{6}}$	$=$	su(5)	$\mathbf{10} \oplus \overline{\mathbf{10}}$	$[6 : \mathrm{sp}(5)]$
7.	sp(2)	$12 \cdot \mathbf{4}$	$=$	su(2)		$[4 : E_7]$
8.	su(4)	$2 \cdot \mathbf{6} \oplus 6 \cdot \mathbf{4} \oplus 6 \cdot \overline{\mathbf{4}}$	$=$	su(3)	$2 \cdot \mathbf{3} \oplus 2 \cdot \overline{\mathbf{3}}$	$[4 : E_7]$
9.	so(7)	$6 \cdot \mathbf{8} \oplus 4 \cdot \mathbf{7}$	$=$	G_2	$4 \cdot \mathbf{7}$	$[4 : E_7]$
10.	so(8)	$6 \cdot \mathbf{8} \oplus 4 \cdot \mathbf{8}' \oplus 2 \cdot \mathbf{8}''$	$=$	so(7)	$6 \cdot \mathbf{8}$	$[4 : E_7]$
11.	so(8)	$6 \cdot \mathbf{8} \oplus 6 \cdot \mathbf{8}'$	$=$	G_2		$[4 : E_7] \oplus [4 : E_7]$
12.	sp(2)	$6 \cdot \mathbf{5}$	$=$	su(2)		$[4 : \mathrm{sp}(3) \oplus \mathrm{su}(2)]$
13.	sp(2)	$4 \cdot \mathbf{4} \oplus 4 \cdot \mathbf{5}$	$=$	su(2)	$3 \cdot \mathbf{2}$	$[4 : \mathrm{sp}(3) \oplus \mathrm{su}(2)]$
14.	su(4)	$\mathbf{10} \oplus \overline{\mathbf{10}} \oplus 2 \cdot \mathbf{4} \oplus 2 \cdot \overline{\mathbf{4}}$	$=$	su(3)	$\mathbf{3} \oplus \overline{\mathbf{3}}$	$[4 : \mathrm{sp}(3) \oplus \mathrm{su}(2)]$
15.	su(3)	$6 \cdot \mathbf{3} \oplus 6 \cdot \overline{\mathbf{3}}$	$=$	su(2)	$2 \cdot \mathbf{2}$	$[3 : E_6]$
16.	su(4)	$4 \cdot \mathbf{6} \oplus 4 \cdot \mathbf{4} \oplus 4 \cdot \overline{\mathbf{4}}$	$=$	sp(2)	$6 \cdot \mathbf{4}$	$[3 : E_6]$
17.	su(3)	$\mathbf{3} \oplus \overline{\mathbf{3}} \oplus \mathbf{6} \oplus \overline{\mathbf{6}}$	$=$	su(2)	$n \cdot \mathbf{2}$	$[3 : \mathfrak{h}]$

- predicted dualities with one marginal operator

Examples of Duality and Results (2 Marginal Operators)

	$\mathfrak{g}$	w/ $\mathbf{r}$	$=$	$\widetilde{\mathfrak{g}}$	w/ $\widetilde{\mathbf{r}}$	$\oplus$ SCFT$[d:\mathfrak{h}]$
18.	su(2) $\oplus$ su(3)	$2{\cdot}(\mathbf{2},\mathbf{1}) \oplus (\mathbf{2},\mathbf{3}{\oplus}\overline{\mathbf{3}}) \oplus 4{\cdot}(\mathbf{1},\mathbf{3}{\oplus}\overline{\mathbf{3}})$	$=$	su(2) $\oplus$ su(2)	$2{\cdot}(\mathbf{2},\mathbf{1}) \oplus 2{\cdot}(\mathbf{1},\mathbf{2})$	$[3:E_6]$
19.	su(2) $\oplus$ sp(2)	$2{\cdot}(\mathbf{2},\mathbf{4}) \oplus 8{\cdot}(\mathbf{1},\mathbf{4})$	$=$	su(2) $\oplus$ su(2)		$[4:E_7]$
20.	su(2) $\oplus$ sp(2)	$3{\cdot}(\mathbf{2},\mathbf{1}) \oplus (\mathbf{2},\mathbf{5}) \oplus 4{\cdot}(\mathbf{1},\mathbf{5})$	$=$	su(2) $\oplus$ su(2)	$3{\cdot}(\mathbf{2},\mathbf{1})$	[4 : sp(3)$\oplus$su(2)]
21.	su(2) $\oplus$ G_2	$(\mathbf{2},\mathbf{1}) \oplus (\mathbf{2},\mathbf{7}) \oplus 6{\cdot}(\mathbf{1},\mathbf{7})$	$=$	su(2) $\oplus$ su(2)	$(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$	[6 : sp(5)]
22.	su(3) $\oplus$ su(3)	$2{\cdot}(\mathbf{3},\overline{\mathbf{3}}) \oplus 2{\cdot}(\overline{\mathbf{3}},\mathbf{3})$	$=$	su(2) $\oplus$ su(3)	$2{\cdot}(\mathbf{2},\mathbf{1})$	$[3:E_6]$
23.	su(3) $\oplus$ su(3)	$(\mathbf{3}{\oplus}\overline{\mathbf{3}},\mathbf{3}{\oplus}\overline{\mathbf{3}})$	$=$	su(2) $\oplus$ su(3)	$2{\cdot}(\mathbf{2},\mathbf{1})$	$[3:E_6]$
24.	su(3) $\oplus$ su(3)	$3{\cdot}(\mathbf{3}{\oplus}\overline{\mathbf{3}},\mathbf{1}) \oplus (\mathbf{3},\overline{\mathbf{3}}) \oplus (\overline{\mathbf{3}},\mathbf{3})$ $\oplus 3{\cdot}(\mathbf{1},\mathbf{3}{\oplus}\overline{\mathbf{3}})$	$=$	su(2) $\oplus$ su(3)	$2{\cdot}(\mathbf{2},\mathbf{1})$ $\oplus 3{\cdot}(\mathbf{1},\mathbf{3}{\oplus}\overline{\mathbf{3}})$	$[3:E_6]$
25.	su(3) $\oplus$ sp(2)	$(\mathbf{3}{\oplus}\overline{\mathbf{3}},\mathbf{1}) \oplus (\mathbf{3}{\oplus}\overline{\mathbf{3}},\mathbf{5})$	$=$	su(2) $\oplus$ sp(2)	$2{\cdot}(\mathbf{2},\mathbf{1})$	$[3:E_6]$
			$=$	su(3) $\oplus$ su(2)	$(\mathbf{3}{\oplus}\overline{\mathbf{3}},\mathbf{1})$	[4 : sp(3)$\oplus$su(2)]
26.	su(3) $\oplus$ sp(2)	$2{\cdot}(\mathbf{3}{\oplus}\overline{\mathbf{3}},\mathbf{1}) \oplus (\mathbf{3}{\oplus}\overline{\mathbf{3}},\mathbf{4}) \oplus 6{\cdot}(\mathbf{1},\mathbf{4})$	$=$	su(2) $\oplus$ sp(2)	$2{\cdot}(\mathbf{2},\mathbf{1}) \oplus 6{\cdot}(\mathbf{1},\mathbf{4})$	$[3:E_6]$
			$=$	su(3) $\oplus$ su(2)	$2{\cdot}(\mathbf{3}{\oplus}\overline{\mathbf{3}},\mathbf{1})$	$[4:E_7]$
27.	sp(2) $\oplus$ sp(2)	$2{\cdot}(\mathbf{5},\mathbf{1}) \oplus (\mathbf{5},\mathbf{4}) \oplus 7{\cdot}(\mathbf{1},\mathbf{4})$	$=$	su(2) $\oplus$ sp(2)	$7{\cdot}(\mathbf{1},\mathbf{4})$	[4 : sp(3)$\oplus$su(2)]
			$=$	sp(2) $\oplus$ su(2)	$2{\cdot}(\mathbf{5},\mathbf{1})$	$[4:E_7]$
28.	sp(2) $\oplus$ sp(2)	$4{\cdot}(\mathbf{4},\mathbf{1}) \oplus 2{\cdot}(\mathbf{4},\mathbf{4}) \oplus 4{\cdot}(\mathbf{1},\mathbf{4})$	$=$	su(2) $\oplus$ sp(2)	$4{\cdot}(\mathbf{1},\mathbf{4})$	$[4:E_7]$
29.	sp(2) $\oplus$ G_2	$5{\cdot}(\mathbf{4},\mathbf{1}) \oplus (\mathbf{4},\mathbf{7}) \oplus 4{\cdot}(\mathbf{1},\mathbf{7})$	$=$	su(2) $\oplus$ G_2	$4{\cdot}(\mathbf{1},\mathbf{7})$	$[4:E_7]$
			$=$	sp(2) $\oplus$ su(2)	$5{\cdot}(\mathbf{4},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$	[6 : sp(5)]

- predicted dualities with two marginal operators

Examples of Duality and Results (New SCFT's)

d	$\mathfrak{h}$	$k_{\mathfrak{h}}$	$3/2 \cdot k_R$	$48 \cdot a$	$\mathbb{Z}_2$ anomaly?
6	E_8	12	124	190	no
6	sp(5)	7	98	164	yes
4	E_7	8	76	118	no
4	sp(3) $\oplus$ su(2)	5 $\oplus$ 8	58	100	yes $\oplus$ no
3	E_6	6	52	82	no
3	$2 \leq \text{rank}(\mathfrak{h}) \leq 6$	$(8-n)/\text{I}_{\text{su}(2)\hookrightarrow\mathfrak{h}}$	38-2n	68-2n	?

- From arguments found in 0712.2028 we can restrict rank($\mathfrak{h}$) $= 2$ which requires $n = 2$ in order to match the flavor symmetries.

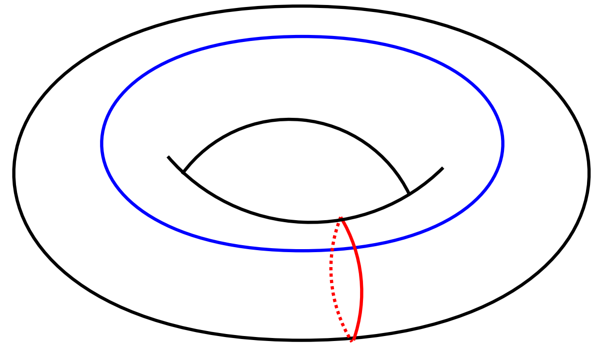
- The flavor central charges, $k_{\mathfrak{h}}$, were confirmed for E_6, E_7, and E_8 through an F-theory calculation by Aharony and Tachikawa in 0711.4532.

IV Seiberg-Witten Theory

The physics is encoded by:

- The Seiberg-Witten curve:
  $y^2 = x^3 + f(u, m_i)x + g(u, m_i)$

- and the Seiberg-Witten 1-form: λ_{SW}
  $\partial_u \lambda_{SW} = \frac{\mathrm{d}x}{y} + \partial_x(\star)\mathrm{d}x.$



The charged states of the theory are encoded by:

- u(1) charges of a physical state are given by the homology class of a cycle, $\gamma = n_e[\alpha] + n_m[\beta]$ (when $m_i = 0$).

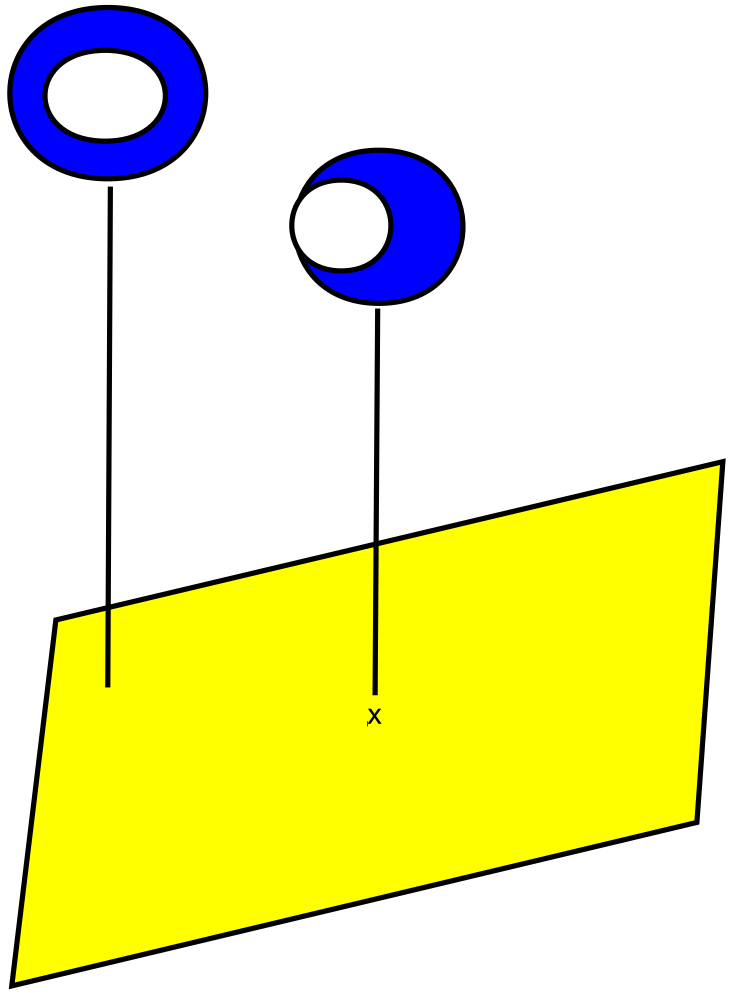
- These states have central charge, Z $= \oint_\gamma \lambda_{SW}$.

Seiberg-Witten Theory(Singularities)

The singularities of the Seiberg-Witten curve:

- are located at $\Delta = 4 \cdot f^3 + 27 \cdot g^2 = 0$.
- If $m_i = 0$ then $\Delta \sim u^n$.
- The singularities physically correspond to a breakdown of the low-energy description $\Rightarrow$ charged states are becoming massless at this point in moduli space.

Seiberg-Witten Theory(Singularities with $m_i = 0$)



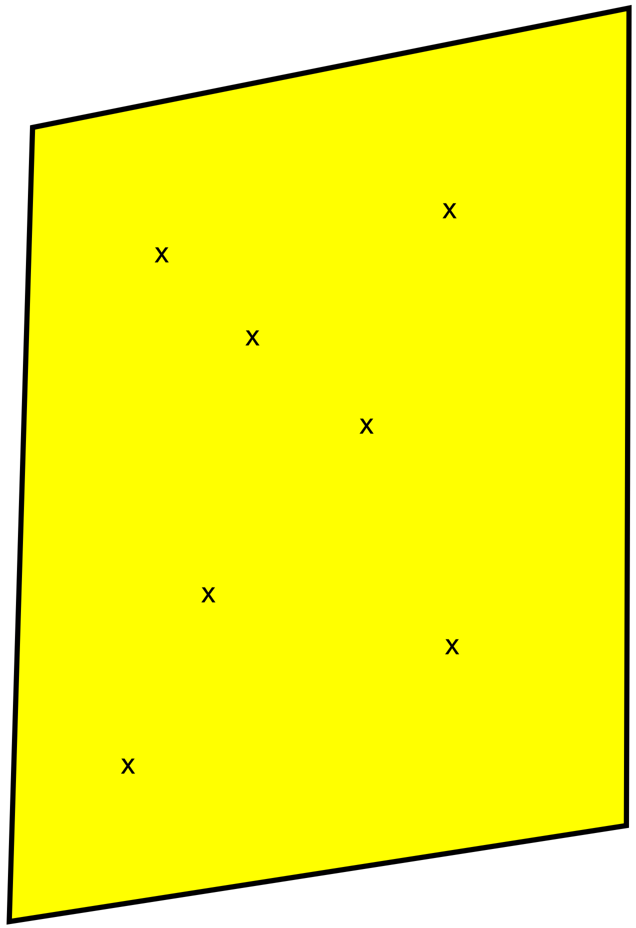
Seiberg-Witten Theory($m_i \neq 0$)

- When mass parameters are turned on they appear in the curve in the form of invariants of the weyl group of the flavor symmetry.

- $\Delta = u^n + P_{D(u)}(\{m_i\})u^{n-1} + ... + P_{nD(u)}(\{m_i\})$

- The factorization of Δ is related to the flavor symmetry group through the fact that different flavor symmetries $\leftrightarrow$ different factorizations of Δ.

Seiberg-Witten Theory(Singularities with $m_i \neq 0$)



V The Kodaira Classification

- Kodaira classified the degenerations of holomorphic families of elliptic curves over one variable, u.

- The classification is of singularities that do not degenerate the holomorphic 1-form at the singularity.

- Fixing the holomorphic 1-form, $\omega = \frac{\mathrm{d}x}{y}$, and requiring the singularities occur as $u \to 0$ specifies the curves exactly up to overall rescalings of u.

The Kodaira Classification

Recall:

- $y^2 = x^3 + f(u)x + g(u)$
- $\partial_u \lambda_{SW} = \frac{\mathrm{d}x}{y} + \partial_x(\star)\mathrm{d}x$
- $\mathrm{Z} = \oint_\gamma \lambda_{SW}$

It is easy to reproduce Kodaira's classification by a little algebra. For details see sections 2.2 & 2.3 of hep-th/0504070 by Philip Argyres, Michael Crescimanno, Alfred Shapere, and JRW.

The Kodaira Classification

name	curve	$\Delta_x \propto$	$D(u)$	$D(x)$
E_8	$y^2 = x^3 + 2u^5$	u^{10}	6	10
E_7	$y^2 = x^3 + u^3x$	u^9	4	6
E_6	$y^2 = x^3 + u^4$	u^8	3	4
D_4	$y^2 = x^3 + 3\tau u^2 x + 2u^3$	$u^6(\tau^3 + 1)$	2	2
H_3	$y^2 = x^3 + u^2$	u^4	3/2	1
H_2	$y^2 = x^3 + ux$	u^3	4/3	2/3
H_1	$y^2 = x^3 + u$	u^2	6/5	2/5
$D_{n>4}$	$y^2 = x^3 + 3ux^2 + 4\Lambda^{-2(n-4)}u^{n-1}$	$u^{n+2}(1 + \Lambda^{-2(n-4)}u^{n-4})$	2	2
$A_{n\geq 0}$	$y^2 = (x-1)(x^2 + \Lambda^{-(n+1)}u^{n+1})$	$u^{n+1}(1 + \Lambda^{-(n+1)}u^{n+1})$	1	0

(1)

- The result is two infinite series and seven "exceptional" curves.
- Λ is the UV strong coupling scale and τ is the marginal gauge coupling.

The Kodaira Classification (Complex Deformations)

- The general mass deformations of these curves correspond to complex structure deformations that are subleading singularities as $u \to 0$.

		$\mathfrak{f}$
E_8	$y^2 = x^3 + x(M_2u^3 + M_8u^2 + M_{14}u + M_{20}) + (2u^5 + M_{12}u^3 + M_{18}u^2 + M_{24}u + M_{30})$	E_8
E_7	$y^2 = x^3 + x(u^3 + M_8u + M_{12}) + (M_2u^4 + M_6u^3 + M_{10}u^2 + M_{14}u + M_{18})$	E_7
E_6	$y^2 = x^3 + x(M_2u^2 + M_5u + M_8) + (u^4 + M_6u^2 + M_9u + M_{12})$	E_6
D_4	$y^2 = x^3 + x(3\tau u^2 + M_2u + M_4) + (2u^3 + \widetilde{M}_4u + M_6)$	so(8)
H_3	$y^2 = x^3 + x(M_{1/2}u + M_2) + (u^2 + M_3)$	u(3)
H_2	$y^2 = x^3 + x(u) + (M_{2/3}u + M_2)$	u(2)
H_1	$y^2 = x^3 + x(M_{4/5}) + (u)$	u(1)
$D_{n>4}$	$y^2 = x^3 + 3ux^2 + \Lambda^{-(n-4)}\widetilde{M}_nx + 4\Lambda^{-2(n-4)}(u^{n-1} + M_2u^{n-2} + \cdots + M_{2n-2})$	so($2n$)
$A_{n\geq 0}$	$y^2 = (x-1)(x^2 + \Lambda^{-(n+1)}[u^{n+1} + M_2u^{n-1} + M_3u^{n-2} + \cdots + M_{n+1}])$	su($n+1$)

(2)

Kodaira Classification(The $A_{n>0}$ series)

- The curve shown corresponds to a u(1) gauge theory with $n+1$ hypermultiplets all of the same charge, ± 1.

- The beta function determines the form of the singularity. Let there be n_a equal mass hypermultiplets with charge $\pm r_a$ then $b = \Sigma_a n_a r_a^2$.

- $b = n + 1 \to A_n$ singularity.

- $b = \Sigma_a n_a r_a^2 \to \oplus_a \text{u}(n_a)$ flavor symmetry.

- Since $b > 0$ these theories are all IR free.

- This is an example of the theme, singularity $\Leftrightarrow$ gauge group.

The Kodaira Classification(The D_n series)

- The curve (for $n > 4$) written corresponds to an su(2) gauge theory with $2n$ half-hypers in the fundamental representation $\Rightarrow b = 2(n-4)$.

- Again $b > 0$ so all these theories are IR free.

- There are two types of representations for su(2), the real $\mathbf{2r+1}$ and the pseudoreal $\mathbf{2s}$.

- To avoid anomalies we must have $2n_r$ of each real representation and any number m_s of the pseudoreal such that $\frac{1}{3}\Sigma_s m_s s(4s^2-1)$ is even.

- $b = \frac{4}{3}\Sigma_r n_r r(r+1)(2r+1) + \frac{1}{3}\Sigma_s m_s s(4s^2-1) - 8$

- The flavor symmetry that corresponds to this value of the beta function is $\oplus_r \text{sp}(n_r) \oplus_s \text{so}(m_s)$.

The Kodaira Classification(Vanishing beta function)

There are two ways to make $b = 0$ for the su(2) beta function.

- The first case is $m_1 = 8$ and all other zero. This has a flavor symmetry of so(8).

- This curve is the fully mass deformed D_4 curve.

- The second case is $n_1 = 1$ and all other zero. This has a flavor symmetry of sp(1) and enhances the susy to $N = 4$.

- The curve for the second case is $y^2 = \prod_i (x - e_i u - e_i^2 M_2)$.

- $\Delta = \prod_{i<j}(e_i - e_j)^2 (u + (e_i + e_j) M_2)^2$

The Kodaira Classification(Asymptotically free (or AF) theories)

These come from looking at su(2) gauge theories with $b < 0$.

- If we put in only fundamental matter: $b = m_1 - 8$.
- m_1 is the number of half-hypers and to avoid anomalies m_1 must be even $\Rightarrow m_1 = 2, 4, 6$.
- When all the masses are taken to be the same we get the $H_{1,2,3}$ mass deformed curves, respectively.

The Kodaira Classification($E_{6,7,8}$ mass deformations)

The $E_{6,7,8}$ curves correspond to strongly interacting fixed points.

- There existence was suggested from stringy constructions.

- The maximal mass deformations were worked out by Minahan and Nemeschansky in:
  Nuclear Physics B 482 (1996) 142-152 and
  Nuclear Physics B 489 (1997) 24-46.

- Evidence for the existence of new mass deformations was found by Philip Argyres & JRW in 0712.2028.

VI Central Charges and Curves

Shapere and Tachikawa 0804.1957

The twisted version of Seiberg-Witten theory relates the anomalies and central charges to:

- The mass dimension of the vev on moduli space,

- the # of neutral hypermultiplets on moduli space and

- the # of singular points of the Seiberg-Witten curve.

Central Charges and Curves (Twisted PI)

$\int [\mathrm{d}u][\mathrm{d}q] A^{\chi} B^{\sigma} C^{n} \mathrm{e}^{-S_{low-energy}}$

- χ is the Euler characteristic.

- σ is the signature.

- n is an instanton number.

- $A^2 = det[\frac{\partial u_i}{\partial a_j}]$

- $B^8 = Radical[\Delta]$

Central Charges and Curves (master equations)

- The scaling behavior of the measure determines the R-charge of the states becoming massless at a singularity.

- The normalization is: $R(\star) = 2 \cdot D(\star)$.

- $48 \cdot a = 12 \cdot R(A) + 8 \cdot R(B) + 10 \cdot r + 2 \cdot \mathrm{h}$

- $24 \cdot c = 8 \cdot R(B) + 4 \cdot r + 2 \cdot \mathrm{h}$

- $4(2 \cdot a - c) = 2 \cdot R(A) + r = \sum_{i=1}^{r} 2 \cdot (d_i - 1) + r = \sum_{i=1}^{r} (2 \cdot d_i - 1)$

$r \equiv$ the complex dimension of moduli space

$\mathrm{h} \equiv$ the # of massless neutral hypermultiplets

Central Charges and Curves ($r = 1$)

- $R(A) = d - 1$
- $R(B) = \frac{1}{4} \cdot Z \cdot d$
- $Z \equiv$ The # of singular points of the Seiberg-Witten curve.
- $24 \cdot c = 2 \cdot Z \cdot d + 4 + 2 \cdot \mathsf{h}$
- $k_{\mathfrak{h}} = 2 \cdot d - \mathsf{h}$

Central Charges and Curves(the unknown solution)

- $15 = 3 \cdot Z + h$

- $\frac{6}{I_{\mathrm{su}(2) \hookrightarrow \mathfrak{h}}} = 6 - h$

- The only rank 2 Lie Algebras are su(2) $\oplus$ su(2), su(3), sp(2), and G_2

Central Charges and Curves (Results)

d	$\mathfrak{h}$	Z	$2 \cdot \mathsf{h}$	rep.'s
6	E_8	10	0	-
6	sp(5)	7	10	$\mathbf{10}$(s)
4	E_7	9	0	-
4	sp(3) $\oplus$ su(2)	6	$(6, 0)$	$\mathbf{6} \oplus \mathbf{1}$(s)
3	E_6	8	0	-
3	rank($\mathfrak{h}$) $= 2$	$4, 5$	$6, 0$	?

Since there are no neutral hypermultiplets on the LHS of the equivalence $\Rightarrow$ the neutral hypermultiplets on the RHS must be charged under the flavor symmetry.

VII $\mathbb{Z}_2$-obstruction Revisited

- There is a $\mathbb{Z}_2$-obstruction for the sp(5) and sp(3) factors.

- This obstruction comes from the neutral hypermultiplet charged in a pseudoreal representation.

- Consider our old example in this new light:

$$G_2 \text{ w/ } 8 \cdot \mathbf{7} \simeq \text{su}(2) \text{ w/ } (\mathbf{2} \oplus \text{SCFT}[6 : \text{sp}(5)])$$
$$\text{su}(2) \oplus \text{sp}(4) \subset \text{sp}(5)$$
$$(\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8}) = \mathbf{10}$$

VIII Constructing New Seiberg-Witten Curves

- The work of Shapere and Tachikawa specifies possible forms of the discriminant of the Seiberg-Witten curves.

- The discriminants are determined by partitioning the total order of the singularity at $m_i = 0$ into a Z-tuple of integers.

Constructing New Seiberg-Witten Curves

There are 4 singular points and 8 singularities.

- $\Delta \sim (u + ...)^5 (u^3 + ...)$
- $\Delta \sim (u + ...)^4 (u + ...)^2 (u^2 + ...)$
- $\Delta \sim (u + ...)^3 (u + ...)(u^2 + ...)^2$
- $\Delta \sim (u^2 + ...)^3 (u^2 + ...)$
- $\Delta \sim (u^4 + ...)^2$

A systematic search for su(3) reveals 2 solutions. We need to carryout a systematic search for su(2) $\oplus$ su(2), G_2, and sp(2).

Constructing New Seiberg-Witten Curves (1^{st} consistent su(3) solution)

- $y^2 = x^3 + 3N_2 x[u^2 + (1+\nu)N_2^3 + N_3^2] + [u^4 + u^2((1+2\nu)N_2^3 + 2N_3^2) + \nu(1+\nu)N_2^6 + (1+2\nu)N_2^3 N_3^2 + N_3^4]$

- $\Delta = -27[u^2 + (1+\nu)N_2^3 + N_3^2]^2[u^2 + (2+\nu)N_2^3 + N_3^2]^2$

- Upon constructing the SW 1-form for this curve we find that it is impossible.

Constructing New Seiberg-Witten Curves (2^{nd} consistent su(3) solution)

- $y^2 = x^3 + u[3N_2x(u - 4N_3) + u^3 - 12u^2N_3 - u(N_2^3 - 48N_3^2) - 64N_3^3]$

- $\Delta = -27u^2[u^3 - 12u^2N_3 + u(N_2^3 + 48N_3^2) - 64N_3^3]^2$

- When we compute the SW 1-form we find that it is identical zero. Therefore this is not a valid solution.

Constructing New Seiberg-Witten Curves

There are 5 singular points and 8 singularities.

- $\Delta \sim (u + ...)^4 (u^4 + ...)$

- $\Delta \sim (u + ...)^3 (u + ...)^2 (u^3 + ...)$

- $\Delta \sim (u^3 + ...)^2 (u^2 + ...)$

We need to carryout a systematic search for su(2) $\oplus$ su(2), su(3), G_2, and sp(2).

Constructing New Seiberg-Witten Curves (sp(3) $\oplus$ su(2))

There are 6 singular points and 9 singularities.

- $\Delta \sim (u + ...)^4(u^5 + ...)$

- $\Delta \sim (u + ...)^3(u + ...)^2(u^4 + ...)$

- $\Delta \sim (u^3 + ...)^2(u^3 + ...)$

A systematic search reduces the problem to solving on the order of 800 polynomial relationships amongst 160 unknowns.

Constructing New Seiberg-Witten Curves (sp(5))

There are 7 singular points and 10 singularities

- $\Delta \sim (u + ...)^4 (u^6 + ...)$
- $\Delta \sim (u + ...)^3 (u + ...)^2 (u^5 + ...)$
- $\Delta \sim (u^3 + ...)^2 (u^4 + ...)$

A systematic search was not attempted for this case because of the outcome found on the previous slide.

IX Isogenies

An isogeny is a many-to-one holomorphic map that preserves the holomorphic 1-form. There are three traditional presentations of elliptic curves which are related by isogenies.

- Legendre: $\eta^2 = \xi^3 + f\xi + g$
- Jacobi: $y^2 = x^4 + \alpha x^2 + \beta$
- Hessian: $\gamma = y^3 + \delta xy + x^3$

Where f, g, α, β, γ, and δ are all functions of u.

Isogenies(2-isogeny)

The map from the Jacobi form to the Legendre form is a 2-isogeny.

- $x = (\xi - \frac{1}{3}\alpha)^{\frac{1}{2}}$
- $y = \eta(\xi - \frac{1}{3}\alpha)^{-\frac{1}{2}}$
- $\Delta = \beta^2(\alpha^2 - 4\beta)$
- The condition for a curve to have a 2-isogeny is that $D(\beta) = kD(u)$ where $k \in \mathbb{Z}^+$.

The H_2, D_4, and E_7 curves have a 2-isogenous form. The H_2 isogenous curve can only have a u(1) flavor symmetry.

Isogenies(2-isogeny of D_4)

- $\alpha = \tau u + M_2$
- $\beta = u^2 + M_4$
- $\Delta = (u^2 + M_4)^2((\tau^2 - 4)u^2 + 2\tau M_2 u + (M_2^2 - 4M_4))$
- If we take the special case $M_4 = \frac{1}{4-\tau^2} M_2^2$ then we get

  $\Delta = (\tau^2 - 4)(u + \frac{\tau}{\tau^2-4} M_2)^2 (u - (\tau^2 - 4)^{-\frac{1}{2}} M_2)^2 (u + (\tau^2 - 4)^{-\frac{1}{2}} M_2)^2$

  This is the same discriminant as the $N = 4$ solution.

Isogenies(2-isogeny of E_7)

- $\alpha = M_2 u + M_6$
- $\beta = u^3 + M_8 u + M_{12}$
- By comparing the dimensions of the complex parameters to the dimensions of the Casimirs of Lie Algebras the maximal flavor symmetry is F_4.
- A systematic computation of the SW 1-form still needs to be performed to see what are the possible flavor symmetries.

Isogenies(3-isogeny)

The map from the Hessian form to the Legendre form is a 3-isogeny.

- $x = -(\xi - \frac{1}{12}\delta^2)(\eta + \frac{1}{2}(\delta\xi - \frac{1}{12}\delta^3 + \gamma))^{-\frac{1}{3}}$

- $y = (\eta + \frac{1}{2}(\delta\xi - \frac{1}{12}\delta^3 + \gamma))^{\frac{1}{3}}$

- $\Delta = \frac{1}{16}\gamma^3(\delta^3 - 27\gamma)$

- The condition for a curve to have a 3-isogeny is the same as a for a 2-isogeny $D(\gamma) = kD(u)$ where $k \in \mathbb{Z}^+$.

The H_3 and E_6 curves have a 3-isogenous form. The H_3 isogenous curve can only have a u(1) flavor symmetry.

Isogenies(3-isogeny of E_6)

- $\delta = M_2$

- $\gamma = u^2 + M_6$

- By explicitly computing the Seiberg-Witten 1-form we find that the flavor symmetry of this curve is G_2.

- The discriminant has $Z = 4$ but it is hard to see how 6 neutral half-hypers could fit into a representation of G_2.

- Try to construct Seiberg-Witten curves for the sp(3) $\oplus$ su(2) and sp(5) flavor symmetries. Systematic searches are plagued with technical difficulties.

- Carryout the remaining systematic searches for the rank 2 flavor symmetry solutions of the E_6 singularity.

- Try to determine the Seiberg-Witten 1-forms and flavor symmetries for the supposed F_4 mass deformation of the E_7 singularity.

- Try to better understand the relationship between isogenies and sub-maximal mass deformations.